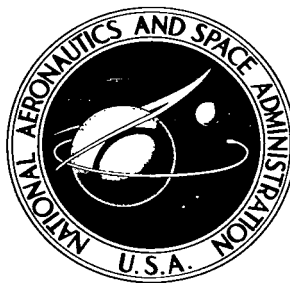


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**ON THE APPLICATION OF PFAFF'S  
METHOD IN THE THEORY OF VARIATION  
OF ASTRONOMICAL CONSTANTS**

*by Peter Musen*

*Goddard Space Flight Center  
Greenbelt, Maryland*



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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# ON THE APPLICATION OF PFAFF'S METHOD IN THE THEORY OF VARIATION OF ASTRONOMICAL CONSTANTS

by

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## SUMMARY

Cartan's integral invariant is taken as a foundation of the theory of variation of astronomical parameters. The differential equations for the general perturbations are obtained as the first system of Pfaffian equations associated with the linear differential form appearing in the integral invariant.

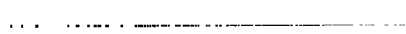
The equations for general perturbations of the Gibbsian unit vectors, of the Gibbsian rotation vector and of Euler's parameters are derived. The utilization of the Gibbsian rotation vector represents an extension of Strömberg's and of the author's ideas on special perturbations to the problems of general perturbations. Euler's parameters find their application in Hansen's lunar theory.

The case of redundant elements is treated by introducing constraints and Lagrangian multipliers. Cartan's integral invariant expressed in terms of vectorial kinematic elements represents a powerful tool in the search for new sets of elements and it leads to differential equations having a compact form, which is convenient for the programming and use of electronic machines.



## CONTENTS

Summary .....	i
Notations .....	iv
INTRODUCTION .....	1
PFAFFIAN EQUATIONS OF MOTION WITH REDUNDANT COORDINATES .....	2
PFAFFIAN EXPRESSION FOR PLANETARY MOTION .....	4
EQUATIONS FOR VARIATION OF VECTORIAL ELEMENTS .....	6
EQUATIONS FOR VARIATION OF EULER'S PARAMETERS...	19
CONCLUSION .....	23
References .....	24



## NOTATIONS

$m_i$  = the mass of the  $i$ -th point in the system of points

$r_i$  = the position vector of the  $i$ -th point

$v_i$  = the velocity of the  $i$ -th point in the system of points

$T$  = the kinetic energy

$U$  = the force function

$F = U - T$

$\Omega$  = the disturbing function

$[\Omega]$  = the disturbing function averaged over the orbit of the disturbed body

$\text{grad}_s \phi$  — the gradient of  $\phi$  with respect to  $s$ ;  $\text{grad}_i \phi$  — the gradient of  $\phi$  with respect to  $s_i$

$k$  = the Gaussian constant

$M$  = the mass of the sun

$m$  = the mass of the disturbed body

$\mu = k^2 (M + m)$

$r$  = the position vector of the disturbed body

$r = |r|$

$v$  = the heliocentric velocity of the disturbed body

$l, \omega, i, \Omega, e, a, n$  = the standard elliptic elements of the disturbed body

$u$  = the eccentric anomaly of the disturbed body

$c = r \times v$  = the vector of the angular momentum

$P, Q, R$  = the Gibbsian vectorial elements of the disturbed body

$e = \mu e P$  = the Laplacian vector

$g$  = the Gibbsian rotation vector

$l', \omega', \Omega', \dots, P', Q', R', \dots$

$l'', \omega'', \Omega'', \dots, P'', Q'', R'', \dots$

the elements of disturbing bodies

$r'^0$  = the unit vector in the direction from the sun to the disturbing body

$P = a(1 - e^2)$

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## INTRODUCTION

In treating the problem of general perturbations of planets and satellites it is felt that a general and flexible method is needed which would permit an easy transformation from one system to another and would also facilitate a proper choice of elements under the different circumstances.

The standard way to develop the equations for the general perturbations in elements is based on the computation of the matrix either of Lagrangian or of Poissonian brackets. This computation is relatively simple for the classical elliptic elements, but it can become cumbersome if some algebraic combinations of the classical elements are taken as a new set of elements.

An additional difficulty may arise if, for the sake of the symmetry of the development of the disturbing function, some redundant elements are introduced. As a result, additional constraints appear in the problem. It is necessary to point out, that the presence of these constraints does not necessarily mean that the problem of the determination of the constants of integration will become more complicated. Sometimes the presence of constraints makes this determination easier. Hill's method of general perturbations is a well known example in celestial mechanics where a redundant constant of integration is present.

However, the difficulties associated with the presence of a redundant constant in this particular case should not be generalized to all problems in celestial mechanics. One should not be hesitant to make use of redundant elements if the problem of the determination of constants of integration for all orders of perturbations can be made symmetrical and if the programming can be made more efficient.

In order to have a more general view and to facilitate the search for new types of elements, an application of Pfaff's method is made in this article to the problems of celestial mechanics. This method permits the formation of the equations for the variation of elements in a straightforward manner if these elements are obtained from the classical elements by any type of functional transformation.

Surprisingly, such a direct method has not found wider applications in the theory of variation of astronomical constants. The pioneering work of Bilimovich (Reference 1) stands rather apart from the main stream. Bilimovich succeeded in deducing Milankovich's equations (Reference 2) for the variation of the Laplacian and momentum vectors in a rather simple way. He also deduced the classical equations for the variation of the elliptic elements in a very simple manner.

The existence of Cartan's (Reference 3) integral invariant for dynamical systems is the foundation of the Pfaffian method. Using a hydrodynamical analogy we can say that the theorem of the circulation for an ideal fluid in the phase space is taken as a basis for the theory of perturbations presented herein. The equations for the variation of the elements represent the first system of Pfaffian equations associated with the differential form appearing under the integral sign in the expression for circulation.

In this work we suggest the use of a slightly more general form of Pfaffian equations than in Bilimovich's work by permitting the constraints to be present and by forming the equations with Lagrangian multipliers.

We shall develop the equations for the general perturbations of Gibbsian vectors  $\mathbf{P}$  and  $\mathbf{Q}$  and also of the elements the author has suggested in his article on Strömberg's (Reference 4) perturbations (Reference 5) and in the articles on Hansen's (Reference 6) lunar theory (References 7 and 16).

In recent times electronic equipment is being used more and more for the purpose of developing general perturbations. We might expect that in the near future the utilization of machines in this domain will become even wider. This fact will have an increasingly greater impact on the theoretical thought and thus the search for different types of new elements is in order now.

## PFAFFIAN EQUATIONS OF MOTION WITH REDUNDANT COORDINATES

Let us consider the material system of  $N$  points having the position vectors  $\mathbf{r}_i$  and the velocity vectors  $\mathbf{v}_i$  ( $i = 1, 2, \dots, N$ ). The Pfaffian linear form associated with this system is

$$\phi = \sum_{i=1}^N m_i \mathbf{v}_i \cdot d\mathbf{r}_i - (T - U) dt . \quad (1)$$

A functional transformation,

$$\begin{aligned} \mathbf{v}_i &= \mathbf{v}_i(x_1, x_2, \dots, x_n; q_1, q_2, \dots, q_m) , \\ \mathbf{r}_i &= \mathbf{r}_i(x_1, x_2, \dots, x_n; q_1, q_2, \dots, q_m) , \end{aligned}$$

with the imposed conditions

$$f_p(x_1, x_2, \dots, x_n; q_1, q_2, \dots, q_m) = a_p$$

brings Equation 1 to the form

$$\begin{aligned} \phi = & \sum_{i=1}^n X_i(x_1, x_2, \dots, x_n; q_1, q_2, \dots, q_m) \cdot dx_i \\ & + \sum_{j=1}^m Q_j(x_1, x_2, \dots, x_n; q_1, q_2, \dots, q_m) dq_j + F dt, \end{aligned} \quad (2)$$

where  $x_1, x_2, \dots, x_n$  are vectors in  $k$ -dimensional Euclidean space and  $q_1, q_2, \dots, q_m$  are scalars.

The condition

$$kn + m - s = 6N$$

must be satisfied. In the process of transformation from Equation 1 to Equation 2 any additive total differential can be neglected.

For present purposes we can assume that all constraints are scleronomic. We write the first system of Pfaffian equations, using the notations suggested by Bilimovich, in the form:

$$\sum_{i=1}^n (\text{grad}_i \phi - dX_i) \cdot \delta x_i + \sum_{j=1}^m \left( \frac{\partial \phi}{\partial q_j} - dQ_j \right) \delta q_j + \left( \frac{\partial \phi}{\partial t} - dF \right) \delta t = 0$$

with the additional conditions

$$\sum_{i=1}^n \text{grad}_i f_p \cdot \delta x_i + \sum_{j=1}^m \frac{\partial f_p}{\partial q_j} \delta q_j = 0 \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, m; p = 1, 2, \dots, s)$$

imposed on the variations by the constraints. Introducing Lagrangian multipliers, we have

$$\text{grad}_i \phi - dX_i + \sum_{p=1}^s \lambda_p dt \text{grad}_i f_p = 0 \quad (3)$$

$$\frac{\partial \phi}{\partial q_j} - dQ_j + \sum_{p=1}^s \lambda_p dt \frac{\partial f_p}{\partial q_j} = 0, \quad (4)$$

$$\frac{\partial F}{\partial t} dt - dF = 0. \quad (5)$$

The Lagrangian multipliers  $\lambda_p$  ( $p = 1, 2, \dots, s$ ) can be determined with the help of the equations

$$\sum_{i=1}^n \text{grad}_i f_p \cdot d\mathbf{x}_i + \sum_{j=1}^m \frac{\partial f_p}{\partial q_j} dq_j = 0, \quad (p = 1, 2, \dots, s). \quad (6)$$

The existence of a Cartan integral invariant on the hypersurface

$$f_p = a_p \quad (p = 1, 2, \dots, s)$$

defined in  $(kn + m)$ -dimensional space serves as a basis for deducing the Pfaffian equations.

Sometimes, for the sake of the symmetry, it might be advantageous to increase beyond six the number of the osculating elements. This is done, for example, by Milankovich (Reference 2), by Herrick (Reference 8) and by the author (Reference 9). Then the relations containing the redundant elements can be understood as the scleronomic constraints and Equations 3-6 can be applied.

## PFAFFIAN EXPRESSION FOR PLANETARY MOTION

In this article we deduce Bilimovich's expression for  $\phi$  for disturbed planetary motion using the Eckert-Brouwer (Reference 10) expression for the variation of the position vector of a planet.

Originally the Eckert-Brouwer formula was designed for the purpose of orbit correction. However, experience has shown that this elegant formula can also be used in the vectorial theory of general perturbations as in the present article and in the article published recently by Musen and Carpenter (Reference 11).

We also suggest herein some new sets of elements, canonical and uncanonical, which follow directly from the form of  $\phi$ .

The differential form (Equation 1) in the case of disturbed planetary motion is

$$\phi = \mathbf{v} \cdot d\mathbf{r} - \left( \frac{v^2}{2} - \frac{\mu}{r} - \Omega \right) dt.$$

Making use of the expression for  $d\mathbf{r}$  as obtained by Eckert and Brouwer,

$$d\mathbf{r} = d\psi \times \mathbf{r} + \frac{\mathbf{v}}{n} dl + \frac{\mathbf{r}}{a} da + \left( \frac{r + p - 2a}{ep} \mathbf{r} + \frac{r + p}{ep} \cdot \frac{\mathbf{r} \cdot \mathbf{v}}{a^2 n^2} \mathbf{v} \right) de,$$

we deduce

$$\mathbf{v} \cdot d\mathbf{r} = \mathbf{r} \times \mathbf{v} \cdot d\boldsymbol{\Psi} + \frac{v^2}{n} dl + \mathbf{r} \cdot \mathbf{v} \frac{da}{a} + \frac{\mathbf{r} \cdot \mathbf{v}}{ep} \left[ (\mathbf{r} + \mathbf{p}) \left( 1 + \frac{v^2}{a^2 n^2} \right) - 2\mathbf{a} \right] de . \quad (7)$$

Substituting

$$\mathbf{r} \cdot \mathbf{v} = \sqrt{\mu a} e \sin u , \quad (8)$$

$$\mathbf{r} \times \mathbf{v} = \mathbf{R} \sqrt{\mu a (1 - e^2)} ,$$

$$v^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right) , \quad (9)$$

$$n = \frac{\sqrt{\mu}}{a^{3/2}} ,$$

into Equation 7 we obtain

$$\mathbf{v} \cdot d\mathbf{r} = \frac{\mu}{n} \left( \frac{2}{r} - \frac{1}{a} \right) dl + \left( \sqrt{\frac{\mu}{a}} e da + \sqrt{\mu a} 2 \frac{a}{r} de \right) \sin u + \sqrt{\mu a (1 - e^2)} \mathbf{R} \cdot d\boldsymbol{\Psi} . \quad (10)$$

Differentiating Equation 8 and taking

$$\frac{r}{a} = 1 - e \cos u$$

into account, we have

$$2d(\mathbf{r} \cdot \mathbf{v}) = \left( e \sqrt{\frac{\mu}{a}} da + 2 \sqrt{\mu a} de \right) \sin u + 2 \sqrt{\mu a} \left( 1 - \frac{r}{a} \right) du . \quad (11)$$

Differentiating Kepler's equation we have

$$du = \frac{a}{r} \sin u de + \frac{a}{r} dl . \quad (12)$$

Eliminating  $du$  from Equation 11 by means of Equation 12 we obtain

$$2d(\mathbf{r} \cdot \mathbf{v}) = \left( e \sqrt{\frac{\mu}{a}} da + 2 \sqrt{\mu a} \frac{a}{r} de \right) \sin u + 2 \sqrt{\mu a} \left( \frac{a}{r} - 1 \right) dl .$$

Taking Equation 9 into account we deduce

$$2d(\mathbf{r} \cdot \mathbf{v}) = \left( \frac{v^2}{n} - \sqrt{\mu a} \right) dl + \left( e \sqrt{\frac{\mu}{a}} da + 2 \frac{a}{r} \sqrt{\mu a} de \right) \sin u .$$

Comparing the last equation with Equation 10 we conclude that

$$\mathbf{v} \cdot d\mathbf{r} = \sqrt{\mu a} dl + \sqrt{\mu a (1 - e^2)} \mathbf{R} \cdot d\boldsymbol{\psi} + 2d(\mathbf{r} \cdot \mathbf{v}) \quad (13)$$

Decomposing the infinitesimal rotation  $d\boldsymbol{\psi}$  along the axes  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{R}$  we have

$$d\boldsymbol{\psi} = d\mathbf{P} \cdot \mathbf{QR} + d\mathbf{Q} \cdot \mathbf{RP} + d\mathbf{R} \cdot \mathbf{PQ}$$

and Equation 13 becomes

$$\mathbf{v} \cdot d\mathbf{r} = \sqrt{\mu a} dl + \sqrt{\mu a (1 - e^2)} \mathbf{Q} \cdot d\mathbf{P} + 2d(\mathbf{r} \cdot \mathbf{v}) \quad (14)$$

Neglecting the total differential  $2d(\mathbf{r} \cdot \mathbf{v})$  we have

$$\phi = \sqrt{\mu a} dl + \sqrt{\mu a (1 - e^2)} \mathbf{Q} \cdot d\mathbf{P} + F dt \quad (15)$$

with  $F = \mu/2a + \Omega$ . Equation 15 was deduced by Bilimovich on the basis of a different consideration.

## EQUATIONS FOR VARIATION OF VECTORIAL ELEMENTS

Milankovich has deduced the vectorial equations for the determination of general perturbations of vectors  $\mathbf{c}$  and  $\mathbf{e}$ .

The vectors  $\mathbf{P}$  and  $\mathbf{Q}$  constitute a better choice for the development of the disturbing function and the computation of the position and velocity vectors.

The elements we choose are

$$l, a, e, \mathbf{P} \text{ and } \mathbf{Q}.$$

For reasons of symmetry we take the expression for  $\phi$  in the form

$$\phi = \sqrt{\mu a} dl + \frac{1}{2} \sqrt{\mu a (1 - e^2)} (\mathbf{Q} \cdot d\mathbf{P} - \mathbf{P} \cdot d\mathbf{Q}) + F dt$$

with the three additional conditions

$$\mathbf{P} \cdot \mathbf{P} = 1, \quad \mathbf{Q} \cdot \mathbf{Q} = 1, \quad \mathbf{P} \cdot \mathbf{Q} = 0. \quad (16)$$

The Pfaffian equations (Equations 3-6) for the variation of constants in this particular case become:

$$\text{grad}_{\mathbf{P}} \phi - d \left[ \frac{1}{2} \sqrt{\mu a (1 - e^2)} \mathbf{Q} \right] + (u\mathbf{P} + w\mathbf{Q}) dt = 0, \quad (17)$$

$$\text{grad}_{\mathbf{Q}} \phi + d \left[ \frac{1}{2} \sqrt{\mu a (1 - e^2)} \mathbf{P} \right] + (v\mathbf{Q} + w\mathbf{P}) dt = 0 , \quad (18)$$

$$\frac{\partial \phi}{\partial a} = 0 , \quad (19)$$

$$\frac{\partial \phi}{\partial e} = 0 , \quad (20)$$

$$\frac{\partial \phi}{\partial I} - d \sqrt{\mu a} = 0 , \quad (21)$$

where  $u$ ,  $v$  and  $w$  here are the Lagrangian multipliers associated with the constraints (Equation 16).

After some easy transformations we deduce from Equations 17-21:

$$na^2 \sqrt{1 - e^2} \frac{d\mathbf{P}}{dt} + (A + w) \mathbf{P} + v\mathbf{Q} + \text{grad}_{\mathbf{Q}} F = 0 , \quad (22)$$

$$na^2 \sqrt{1 - e^2} \frac{d\mathbf{Q}}{dt} + (A - w) \mathbf{Q} - u\mathbf{P} - \text{grad}_{\mathbf{P}} F = 0 , \quad (23)$$

$$\frac{1}{2} na \frac{dI}{dt} + \frac{1}{2} na \sqrt{1 - e^2} \mathbf{Q} \cdot \frac{d\mathbf{P}}{dt} + \frac{\partial F}{\partial a} = 0 , \quad (24)$$

$$\mathbf{Q} \cdot \frac{d\mathbf{P}}{dt} = - \mathbf{P} \cdot \frac{d\mathbf{Q}}{dt} = + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial F}{\partial e} , \quad (25)$$

$$\frac{\partial F}{\partial I} - \frac{1}{2} na \frac{da}{dt} = 0 , \quad (26)$$

where

$$A = \frac{1}{4} \left( na \sqrt{1 - e^2} \frac{da}{dt} - \frac{2na^2 e}{\sqrt{1 - e^2}} \frac{de}{dt} \right) . \quad (27)$$

By multiplying Equation 22 by  $\mathbf{P}$  and Equation 23 by  $\mathbf{Q}$  we deduce:

$$A + w = -\mathbf{P} \cdot \text{grad}_{\mathbf{Q}} F ,$$

$$A - w = +\mathbf{Q} \cdot \text{grad}_{\mathbf{P}} F ,$$

$$A = + \frac{1}{2} \mathbf{Q} \cdot \text{grad}_{\mathbf{P}} F - \frac{1}{2} \mathbf{P} \cdot \text{grad}_{\mathbf{Q}} F , \quad (28)$$

$$w = - \frac{1}{2} \mathbf{Q} \cdot \text{grad}_{\mathbf{P}} F - \frac{1}{2} \mathbf{P} \cdot \text{grad}_{\mathbf{Q}} F . \quad (29)$$

Multiplying Equation 22 by  $\mathbf{Q}$  and Equation 23 by  $\mathbf{P}$  and taking Equation 25 into account, we have

$$\mathbf{v} = -\mathbf{Q} \cdot \text{grad}_{\mathbf{Q}} F - \frac{1-e^2}{e} \frac{\partial F}{\partial e}, \quad (30)$$

$$\mathbf{u} = -\mathbf{P} \cdot \text{grad}_{\mathbf{P}} F - \frac{1-e^2}{e} \frac{\partial F}{\partial e}. \quad (31)$$

Substituting Equations 29-31 into Equations 22 and 23 and taking the identity

$$\mathbf{I} = \mathbf{PP} + \mathbf{QQ} + \mathbf{RR}$$

into account, we obtain

$$\frac{d\mathbf{P}}{dt} = -\frac{1}{na^2} \frac{1}{\sqrt{1-e^2}} \mathbf{RR} \cdot \text{grad}_{\mathbf{Q}} F + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \mathbf{Q}, \quad (32)$$

$$\frac{d\mathbf{Q}}{dt} = +\frac{1}{na^2} \frac{1}{\sqrt{1-e^2}} \mathbf{RR} \cdot \text{grad}_{\mathbf{P}} F - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \mathbf{P}. \quad (33)$$

These two last equations are new. Equations 24 and 26 can be written in the classical form:

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial F}{\partial l}, \quad (34)$$

$$\frac{dl}{dt} = -\frac{1-e^2}{na^2 e} \frac{\partial F}{\partial e} - \frac{2}{na} \frac{\partial F}{\partial a}. \quad (35)$$

From Equations 27, 28, and 34 we deduce

$$\frac{de}{dt} = +\frac{1-e^2}{na^2 e} \frac{\partial F}{\partial l} + \frac{\sqrt{1-e^2}}{na^2 e} (\mathbf{P} \cdot \text{grad}_{\mathbf{Q}} F - \mathbf{Q} \cdot \text{grad}_{\mathbf{P}} F). \quad (36)$$

The system Equations 32-36 forms a complete set to be used in the development of perturbations.

From Equations 32 and 33 we see that the constraints (Equation 16) are satisfied. Because of the existence of these constraints the determination of the additive constants of integration associated with Equations 32 and 33 becomes extremely simple. This simplicity justifies in part the choice of  $\mathbf{P}$  and  $\mathbf{Q}$  as the basic elements. Changing the notation, we shall designate any disturbed element  $E$  as a series

$$E = E_0 + E_1 + E_2 + E_3 + \dots,$$

where  $E$  now means the undisturbed value and  $E_k$  are the perturbations of  $k$ -th order. We deduce

from Equation 16:

$$\begin{aligned} \mathbf{P} \cdot \mathbf{P}_1 &= 0 , \\ \mathbf{P} \cdot \mathbf{P}_2 + \frac{1}{2} \mathbf{P}_1^2 &= 0 , \end{aligned} \tag{37}$$

$$\begin{aligned} \mathbf{P} \cdot \mathbf{P}_3 + \mathbf{P}_1 \cdot \mathbf{P}_2 &= 0 , \\ &\dots\dots\dots \\ \mathbf{Q} \cdot \mathbf{Q}_1 &= 0 , \\ \mathbf{Q} \cdot \mathbf{Q}_2 + \frac{1}{2} \mathbf{Q}_1^2 &= 0 , \end{aligned} \tag{38}$$

$$\begin{aligned} \mathbf{Q} \cdot \mathbf{Q}_3 + \mathbf{Q}_1 \cdot \mathbf{Q}_2 &= 0 , \\ &\dots\dots\dots \\ \mathbf{Q} \cdot \mathbf{P}_1 + \mathbf{P} \cdot \mathbf{Q}_1 &= 0 , \\ \mathbf{Q} \cdot \mathbf{P}_2 + \mathbf{P} \cdot \mathbf{Q}_2 + \mathbf{P}_1 \cdot \mathbf{Q}_1 &= 0 , \end{aligned} \tag{39}$$

$$\begin{aligned} \mathbf{Q} \cdot \mathbf{P}_3 + \mathbf{P} \cdot \mathbf{Q}_3 + \mathbf{P}_1 \cdot \mathbf{Q}_2 + \mathbf{P}_2 \cdot \mathbf{Q}_1 &= 0 . \\ &\dots\dots\dots \end{aligned}$$

For  $\mathbf{P}_k$ ,  $\mathbf{Q}_k$  ( $k = 1, 2, \dots$ ) we obtain from Equations 32 and 33 equations of the form:

$$\frac{d\mathbf{P}_k}{dt} = f_k(\mathbf{P}, \mathbf{Q}, a, l, e; \mathbf{P}_1, \mathbf{Q}_1, a_1, l_1, e_1; \dots; \mathbf{P}_{k-1}, \mathbf{Q}_{k-1}, a_{k-1}, l_{k-1}, e_{k-1}) ,$$

$$\frac{d\mathbf{Q}_k}{dt} = \phi_k(\mathbf{P}, \mathbf{Q}, a, l, e; \mathbf{P}_1, \mathbf{Q}_1, a_1, l_1, e_1; \dots; \mathbf{P}_{k-1}, \mathbf{Q}_{k-1}, a_{k-1}, l_{k-1}, e_{k-1}) .$$

Let

$$(\mathbf{P}_k) = \int f_k dt ,$$

$$(\mathbf{Q}_k) = \int \phi_k dt ,$$

where the integration is performed in a formal manner. We can write

$$\mathbf{P}_k = (\mathbf{P}_k) + p_{k1} \mathbf{P} + p_{k2} \mathbf{Q} + p_{k3} \mathbf{R} ,$$

$$\mathbf{Q}_k = (\mathbf{Q}_k) + q_{k1} \mathbf{P} + q_{k2} \mathbf{Q} + q_{k3} \mathbf{R} ,$$

where  $p_{kj}, q_{kj}$  are the constants of integration. The expressions  $(\mathbf{P}_k), (\mathbf{Q}_k)$  contain the secular, the mixed and the purely periodic terms, but there are no constant terms.

Let  $[f]$  designate the constant part in the multiple Fourier series of some function  $f(l, l', l'', \dots)$ . Taking into account

$$[(\mathbf{P}_k)] = [(\mathbf{Q}_k)] = 0 ,$$

we deduce from Equations 37-39:

$$p_{11} = 0 ,$$

$$p_{21} = -\frac{1}{2} [\mathbf{P}_1^2] , \tag{40}$$

$$p_{31} = -[\mathbf{P}_1 \cdot \mathbf{P}_2] ,$$

.....

$$q_{12} = 0 ,$$

$$q_{22} = -\frac{1}{2} [\mathbf{Q}_1^2] , \tag{41}$$

$$q_{32} = -[\mathbf{Q}_1 \cdot \mathbf{Q}_2] ,$$

.....

$$q_{11} = -p_{12} ,$$

$$q_{21} = -p_{22} - [\mathbf{P}_1 \cdot \mathbf{Q}_1] , \tag{42}$$

$$q_{31} = -p_{32} - [\mathbf{P}_1 \cdot \mathbf{Q}_2] - [\mathbf{P}_2 \cdot \mathbf{Q}_1] .$$

.....

We see from Equations 40-42 that in the process of the determination of the perturbations of  $k$ -th order in  $P$  and  $Q$  the three constants  $p_{k2}$ ,  $p_{k3}$ ,  $q_{k3}$  can be considered as independent. The constants  $q_{k1}$  are determined in terms of  $p_{k2}$  and of the constants of the lower orders. The constants  $p_{k1}$  and  $q_{k2}$  are also functions of the constants of the lower orders. Although there are redundant elements there are no redundant constants which cannot be determined easily. For the disturbed position vector  $r$  we have the standard expression

$$r = Pa(\cos u - e) + Qa\sqrt{1-e^2}\sin u,$$

$$u - e \sin u = l,$$

where all symbols now designate the disturbed elements.

In order to verify the correctness of Equations 32, 33, and 36 we shall deduce them from the corresponding classical equations. From

$$P = \begin{Bmatrix} + \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i \\ + \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i \\ + \sin \omega \sin i \end{Bmatrix},$$

$$Q = \begin{Bmatrix} -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i \\ -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i \\ + \cos \omega \sin i \end{Bmatrix},$$

$$R = \begin{Bmatrix} + \sin \Omega \sin i \\ -\cos \Omega \sin i \\ + \cos i \end{Bmatrix},$$

we have

$$\frac{dP}{dt} = Q \frac{d\omega}{dt} + R \sin \omega \frac{di}{dt} + k \times P \frac{d\Omega}{dt}, \quad (43)$$

$$\frac{dQ}{dt} = -P \frac{d\omega}{dt} + R \cos \omega \frac{di}{dt} + k \times Q \frac{d\Omega}{dt}, \quad (44)$$

and as a consequence of the two last equations:

$$\sin i \frac{d\Omega}{dt} = R \cdot \left( -\frac{dP}{dt} \cos \omega + \frac{dQ}{dt} \sin \omega \right), \quad (45)$$

$$\frac{di}{dt} = \mathbf{R} \cdot \left( + \frac{d\mathbf{P}}{dt} \sin \omega + \frac{d\mathbf{Q}}{dt} \cos \omega \right), \quad (46)$$

$$\frac{d\omega}{dt} + \cos i \frac{d\varpi}{dt} = \mathbf{Q} \cdot \frac{d\mathbf{P}}{dt} = -\mathbf{P} \cdot \frac{d\mathbf{Q}}{dt}. \quad (47)$$

From

$$\frac{\partial F}{\partial \mathbf{E}} = \frac{\partial \mathbf{P}}{\partial \mathbf{E}} \cdot \text{grad}_{\mathbf{P}} F + \frac{\partial \mathbf{Q}}{\partial \mathbf{E}} \cdot \text{grad}_{\mathbf{Q}} F,$$

where

$$\mathbf{E} = \omega, \varpi, i,$$

and Equations 43 and 44, we deduce

$$\frac{\partial F}{\partial \omega} = \mathbf{Q} \cdot \text{grad}_{\mathbf{P}} F - \mathbf{P} \cdot \text{grad}_{\mathbf{Q}} F, \quad (48)$$

$$\frac{\partial F}{\partial i} = \sin \omega (\mathbf{R} \cdot \text{grad}_{\mathbf{P}} F) + \cos \omega (\mathbf{R} \cdot \text{grad}_{\mathbf{Q}} F), \quad (49)$$

$$\frac{\partial F}{\partial \varpi} = \mathbf{k} \times \mathbf{P} \cdot \text{grad}_{\mathbf{P}} F + \mathbf{k} \times \mathbf{Q} \cdot \text{grad}_{\mathbf{Q}} F. \quad (50)$$

Substituting

$$\mathbf{k} = (\mathbf{P} \sin \omega + \mathbf{Q} \cos \omega) \sin i + \mathbf{R} \cos i$$

into Equation 50, we obtain

$$-\sin \omega (\mathbf{R} \cdot \text{grad}_{\mathbf{Q}} F) + \cos \omega (\mathbf{R} \cdot \text{grad}_{\mathbf{P}} F) = -\frac{1}{\sin i} \frac{\partial F}{\partial \varpi} + \text{ctg } i \frac{\partial F}{\partial \omega}.$$

From Equations 36, 45-47 and Equations 32, 33, 48-50 we obtain the classical equations:

$$\frac{d\varpi}{dt} = \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial i},$$

$$\frac{di}{dt} = -\frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial \varpi} + \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial \omega},$$

$$\frac{d\omega}{dt} = + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} - \cos i \frac{d\varpi}{dt},$$

$$\frac{de}{dt} = + \frac{1 - e^2}{na^2 e} \frac{\partial F}{\partial t} - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial F}{\partial \omega} .$$

Let the matrix of direction-cosines between  $(P, Q, R)$  and  $(P', Q', R')$  be:

$$\begin{aligned} P \cdot P' &= \alpha_1 , & P \cdot Q' &= \alpha_2 , & P \cdot R' &= \alpha_3 , \\ Q \cdot P' &= \beta_1 , & Q \cdot Q' &= \beta_2 , & Q \cdot R' &= \beta_3 , \\ R \cdot P' &= \gamma_1 , & R \cdot Q' &= \gamma_2 , & R \cdot R' &= \gamma_3 . \end{aligned}$$

The disturbing function depends upon  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$ . Taking into account

$$\begin{aligned} \text{grad}_P \alpha_1 &= \text{grad}_Q \beta_1 = P' , \\ \text{grad}_Q \beta_2 &= \text{grad}_P \alpha_2 = Q' , \\ \text{grad}_Q \alpha_1 &= \text{grad}_Q \alpha_2 = \text{grad}_P \beta_1 = \text{grad}_P \beta_2 = 0 , \end{aligned}$$

we obtain:

$$\text{grad}_P F = \frac{\partial F}{\partial \alpha_1} P' + \frac{\partial F}{\partial \alpha_2} Q' , \quad (51)$$

$$\text{grad}_Q F = \frac{\partial F}{\partial \beta_1} P' + \frac{\partial F}{\partial \beta_2} Q' , \quad (52)$$

$$R \cdot \text{grad}_P F = \frac{\partial F}{\partial \alpha_1} \gamma_1 + \frac{\partial F}{\partial \alpha_2} \gamma_2 ,$$

$$R \cdot \text{grad}_Q F = \frac{\partial F}{\partial \beta_1} \gamma_1 + \frac{\partial F}{\partial \beta_2} \gamma_2 ;$$

and the Equations 32 and 33 become

$$\begin{aligned} \frac{dP}{dt} &= - \frac{1}{na^2 \sqrt{1 - e^2}} \left( \frac{\partial F}{\partial \beta_1} \gamma_1 + \frac{\partial F}{\partial \beta_2} \gamma_2 \right) R + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial F}{\partial e} Q , \\ \frac{dQ}{dt} &= + \frac{1}{na^2 \sqrt{1 - e^2}} \left( \frac{\partial F}{\partial \alpha_1} \gamma_1 + \frac{\partial F}{\partial \alpha_2} \gamma_2 \right) R - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial F}{\partial e} P . \end{aligned}$$

If the frame  $(P', Q', R')$  is a basic reference frame, then the last two equations become:

$$\frac{d\mathbf{P}}{dt} = - \frac{1}{na^2 \sqrt{1-e^2}} \left( \frac{\partial F}{\partial \beta_1} \gamma_1 + \frac{\partial F}{\partial \beta_2} \gamma_2 \right) \mathbf{R} + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \mathbf{Q} + \mathbf{P} \times \boldsymbol{\Psi}' , \quad (53)$$

$$\frac{d\mathbf{Q}}{dt} = + \frac{1}{na^2 \sqrt{1-e^2}} \left( \frac{\partial F}{\partial \alpha_1} \gamma_1 + \frac{\partial F}{\partial \alpha_2} \gamma_2 \right) \mathbf{R} - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \mathbf{P} + \mathbf{Q} \times \boldsymbol{\Psi}' , \quad (54)$$

where

$$\boldsymbol{\Psi}' = \psi'_1 \mathbf{P}' + \psi'_2 \mathbf{Q}' + \psi'_3 \mathbf{R}'$$

is the absolute angular velocity of rotation of the frame  $(\mathbf{P}', \mathbf{Q}', \mathbf{R}')$ , and

$$\psi'_1 = + \sin \omega' \sin i' \frac{d\alpha'}{dt} + \cos \omega' \frac{di'}{dt} ,$$

$$\psi'_2 = + \cos \omega' \sin i' \frac{d\alpha'}{dt} - \sin \omega' \frac{di'}{dt} ,$$

$$\psi'_3 = \cos i' \frac{d\omega'}{dt} + \frac{d\omega'}{dt} .$$

From Equations 53 and 54 we deduce the following system of scalar equations for the variation of the direction-cosines:

$$\frac{d\alpha_1}{dt} = - \frac{1}{na^2 \sqrt{1-e^2}} \gamma_1 \left( \frac{\partial F}{\partial \beta_1} \gamma_1 + \frac{\partial F}{\partial \beta_2} \gamma_2 \right) + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \beta_1 + (\alpha_2 \psi'_3 - \alpha_3 \psi'_2) ,$$

$$\frac{d\alpha_2}{dt} = - \frac{1}{na^2 \sqrt{1-e^2}} \gamma_2 \left( \frac{\partial F}{\partial \beta_1} \gamma_1 + \frac{\partial F}{\partial \beta_2} \gamma_2 \right) + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \beta_2 + (\alpha_3 \psi'_1 - \alpha_1 \psi'_3) ,$$

$$\frac{d\alpha_3}{dt} = - \frac{1}{na^2 \sqrt{1-e^2}} \gamma_3 \left( \frac{\partial F}{\partial \beta_1} \gamma_1 + \frac{\partial F}{\partial \beta_2} \gamma_2 \right) + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \beta_3 + (\alpha_1 \psi'_2 - \alpha_2 \psi'_1) ,$$

and

$$\frac{d\beta_1}{dt} = + \frac{1}{na^2 \sqrt{1-e^2}} \gamma_1 \left( \frac{\partial F}{\partial \alpha_1} \gamma_1 + \frac{\partial F}{\partial \alpha_2} \gamma_2 \right) - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \alpha_1 + (\beta_2 \psi'_3 - \beta_3 \psi'_2) ,$$

$$\frac{d\beta_2}{dt} = + \frac{1}{na^2 \sqrt{1-e^2}} \gamma_2 \left( \frac{\partial F}{\partial \alpha_1} \gamma_1 + \frac{\partial F}{\partial \alpha_2} \gamma_2 \right) - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} \alpha_2 + (\beta_3 \psi'_1 - \beta_1 \psi'_3) ,$$

$$\frac{d\beta_3}{dt} = + \frac{1}{na^2 \sqrt{1-e^2}} \gamma_3 \left( \frac{\partial F}{\partial \alpha_1} \gamma_1 + \frac{\partial F}{\partial \alpha_2} \gamma_2 \right) - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial F}{\partial e} a_3 + (\beta_1 \psi'_2 - \beta_2 \psi'_1) .$$

Substituting Equations 51 and 52 into Equation 36 we obtain

$$\frac{de}{dt} = + \frac{1-e^2}{na^2 e} \frac{\partial F}{\partial l} + \frac{\sqrt{1-e^2}}{na^2 e} \left( \alpha_1 \frac{\partial F}{\partial \beta_1} - \beta_1 \frac{\partial F}{\partial \alpha_1} + \alpha_2 \frac{\partial F}{\partial \beta_2} - \beta_2 \frac{\partial F}{\partial \alpha_2} \right) .$$

Equations 32, 33, and 36 can be applied to the determination of the lunar and solar long period effects in the motion of close satellites. In the case of close satellites the short period terms, depending upon the mean anomaly of the satellite, can be neglected and the remaining portion  $[\Omega]$  of the disturbing function can be developed (References 12 and 13) into the series of polynomials in  $e$  and in

$$\begin{aligned} \alpha &= \alpha_1 \cos f' + \alpha_2 \sin f' , \\ \beta &= \beta_1 \cos f' + \beta_2 \sin f' , \\ \gamma &= \gamma_1 \cos f' + \gamma_2 \sin f' . \end{aligned}$$

Taking into account

$$\begin{aligned} \text{grad}_{\mathbf{P}} \alpha &= \text{grad}_{\mathbf{Q}} \beta = \mathbf{r}'^0 , \\ \text{grad}_{\mathbf{Q}} \alpha &= \text{grad}_{\mathbf{P}} \beta = 0 , \end{aligned}$$

we obtain from Equations 32, 33 and 36

$$\begin{aligned} \frac{d\mathbf{P}}{dt} &= - \frac{\gamma \mathbf{R}}{na^2 \sqrt{1-e^2}} \frac{\partial [\Omega]}{\partial \beta} + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial [\Omega]}{\partial e} \mathbf{Q} , \\ \frac{d\mathbf{Q}}{dt} &= + \frac{\gamma \mathbf{R}}{na^2 \sqrt{1-e^2}} \frac{\partial [\Omega]}{\partial \alpha} - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial [\Omega]}{\partial e} \mathbf{P} , \end{aligned}$$

and

$$\frac{de}{dt} = + \frac{\sqrt{1-e^2}}{na^2 e} \left( \alpha \frac{\partial [\Omega]}{\partial \beta} - \beta \frac{\partial [\Omega]}{\partial \alpha} \right) .$$

A very elegant and convenient development of the disturbing function for close satellites was given by Kaula (Reference 14). The use of the Gibbsian rotation vector  $\mathbf{g}$  (Reference 15) was suggested in an article (Reference 5) on the improvement of Strömberg's theory (Reference 4).

The basic vectors  $\mathbf{P}$  and  $\mathbf{Q}$  can be represented as relatively simple functions of  $\mathbf{g}$  and the introduction of  $\mathbf{g}$  removes the necessity of keeping constraints (Equation 16) as the additional conditions. The components of  $\mathbf{g}$  in the inertial system are

$$\begin{aligned} g_1 &= + \frac{\cos \frac{1}{2} (\omega - \Omega)}{\cos \frac{1}{2} (\omega + \Omega)} \operatorname{tg} \frac{i}{2} , \\ g_2 &= - \frac{\sin \frac{1}{2} (\omega - \Omega)}{\cos \frac{1}{2} (\omega + \Omega)} \operatorname{tg} \frac{i}{2} , \\ g_3 &= + \operatorname{tg} \frac{1}{2} (\omega + \Omega) . \end{aligned}$$

Equation 47 shows that the Pfaffian (Equation 15) can be written as

$$\phi = \sqrt{\mu a} \, dl + \sqrt{\mu a (1 - e^2)} (d\omega + \cos i \, d\Omega) + F \, dt . \quad (55)$$

This form was originally used by Bilimovich. Taking

$$\begin{aligned} \omega - \Omega &= -2 \arctan \frac{g_2}{g_1} , \\ \omega + \Omega &= 2 \arctan g_3 , \end{aligned}$$

and

$$\tan^2 \frac{i}{2} = \frac{g_1^2 + g_2^2}{1 + g_3^2}$$

into account, we deduce from Equation 55

$$\phi = \sqrt{\mu a} \, dl + \frac{2 \sqrt{\mu a (1 - e^2)}}{1 + g^2} \mathbf{s} \cdot d\mathbf{g} + F \, dt , \quad (56)$$

where we put

$$\mathbf{s} = \mathbf{k} + \mathbf{g} \times \mathbf{k} . \quad (57)$$

The Pfaffian equation corresponding to  $\mathbf{g}$  is

$$\text{grad}_{\mathbf{g}} \left\{ \frac{2 \sqrt{\mu a (1 - e^2)}}{1 + \mathbf{g}^2} \cdot \mathbf{s} \cdot \frac{d\mathbf{g}}{dt} \right\} - d \left\{ \frac{2 \sqrt{\mu a (1 - e^2)}}{1 + \mathbf{g}^2} \mathbf{s} \right\} + \text{grad}_{\mathbf{g}} F dt = 0$$

or, after the differentiation is performed,

$$\begin{aligned} & \frac{4na^2 \sqrt{1 - e^2}}{1 + \mathbf{g}^2} \left( \mathbf{k} \times \frac{d\mathbf{g}}{dt} + \frac{\mathbf{s} \mathbf{g} \cdot \frac{d\mathbf{g}}{dt} - \mathbf{g} \mathbf{s} \cdot \frac{d\mathbf{g}}{dt}}{1 + \mathbf{g}^2} \right) \\ & - \left( na \sqrt{1 - e^2} \frac{da}{dt} - \frac{2na^2 e}{\sqrt{1 - e^2}} \frac{de}{dt} \right) \frac{\mathbf{s}}{1 + \mathbf{g}^2} + \text{grad}_{\mathbf{g}} F = 0 . \end{aligned} \quad (58)$$

Taking into account

$$\mathbf{s} \mathbf{g} \cdot \frac{d\mathbf{g}}{dt} - \mathbf{g} \mathbf{s} \cdot \frac{d\mathbf{g}}{dt} = (\mathbf{g} \times \mathbf{s}) \times \frac{d\mathbf{g}}{dt} ,$$

we can write Equation 58 in the form

$$\frac{4na^2 \sqrt{1 - e^2}}{1 + \mathbf{g}^2} \mathbf{h} \times \frac{d\mathbf{g}}{dt} - \left( na \sqrt{1 - e^2} \frac{da}{dt} - \frac{2na^2 e}{\sqrt{1 - e^2}} \frac{de}{dt} \right) \frac{\mathbf{s}}{1 + \mathbf{g}^2} + \text{grad}_{\mathbf{g}} F = 0 , \quad (59)$$

where

$$\mathbf{h} = \mathbf{k} + \frac{\mathbf{g} \times \mathbf{s}}{1 + \mathbf{g}^2} = \frac{\mathbf{k} + \mathbf{g} \times \mathbf{k} + \mathbf{g} \mathbf{g} \cdot \mathbf{k}}{1 + \mathbf{g}^2} . \quad (60)$$

We have from Equation 59:

$$\frac{4na^2 \sqrt{1 - e^2}}{1 + \mathbf{g}^2} \mathbf{s} \times \left( \mathbf{h} \times \frac{d\mathbf{g}}{dt} \right) + \mathbf{s} \times \text{grad}_{\mathbf{g}} F = 0 ,$$

or, after developing the triple product,

$$\mathbf{h} \cdot \mathbf{s} \frac{d\mathbf{g}}{dt} = \mathbf{h} \mathbf{s} \cdot \frac{d\mathbf{g}}{dt} + \frac{1 + \mathbf{g}^2}{4na^2 \sqrt{1 - e^2}} \mathbf{s} \times \text{grad}_{\mathbf{g}} F . \quad (61)$$

From Equations 57 and 60 we have

$$\mathbf{h} \cdot \mathbf{s} = 1 , \quad (62)$$

and Equation 61 becomes

$$\frac{d\mathbf{g}}{dt} = \mathbf{h} \mathbf{s} \cdot \frac{d\mathbf{g}}{dt} + \frac{1 + \mathbf{g}^2}{4na^2 \sqrt{1 - e^2}} \mathbf{s} \times \text{grad}_{\mathbf{g}} F . \quad (63)$$

The Pfaffian equation associated with  $e$  is simply

$$\frac{\partial \phi}{\partial e} = 0$$

or

$$- \frac{2na^2 e}{\sqrt{1-e^2}} \frac{\mathbf{s} \cdot d\mathbf{g}}{1+g^2} + \frac{\partial F}{\partial e} dt = 0. \quad (64)$$

From Equations 63 and 64 we obtain

$$\frac{d\mathbf{g}}{dt} = h \frac{(1+g^2)}{2na^2 e} \frac{\sqrt{1-e^2}}{1+g^2} \frac{\partial F}{\partial e} + \frac{1+g^2}{4na^2 \sqrt{1-e^2}} \mathbf{s} \times \text{grad}_{\mathbf{g}} F. \quad (65)$$

Multiplying Equation 59 by  $h$  and taking Equations 62 and 34 into account, we deduce

$$\frac{de}{dt} = + \frac{1-e^2}{na^2 e} \frac{\partial F}{\partial l} - \frac{(1+g^2)}{2na^2 e} \frac{\sqrt{1-e^2}}{1+g^2} h \cdot \text{grad}_{\mathbf{g}} F. \quad (66)$$

The use of the Gibbsian rotation vector represents an application and an extension of Strömgren's and of the author's ideas from the domain of special perturbations to the domain of general perturbations. The use of the vector  $\mathbf{g}$  becomes especially convenient if the basic reference plane nearly coincides with the osculating orbit plane at the initial epoch. Then the components of  $\mathbf{g}$  become small—of the order of perturbations.

Now let  $a, e, l$  be the disturbed and  $P, Q$  the un-disturbed elements and put

$$\mathbf{r}_0 = P\mathbf{a}(\cos u - e) + Q\mathbf{a}\sqrt{1-e^2} \sin u.$$

$$u - e \sin u = l$$

Writing the matrix of rotation  $\Gamma$ , which transforms  $\mathbf{r}_0$  into the disturbed position vector  $\mathbf{r}$ , in the Gibbsian form

$$\Gamma = I + \frac{2}{1+g^2} [\mathbf{g} \times I + \mathbf{g} \times (\mathbf{g} \times I)] , \quad (67)$$

we have

$$\mathbf{r} = \Gamma \cdot \mathbf{r}_0 ,$$

or, in the developed form,

$$\mathbf{r} = \mathbf{r}_0 + \frac{2}{1 + \mathbf{g}^2} [\mathbf{g} \times \mathbf{r}_0 + \mathbf{g} \times (\mathbf{g} \times \mathbf{r}_0)] ,$$

or

$$\mathbf{r} = \frac{1 - \mathbf{g}^2}{1 + \mathbf{g}^2} \mathbf{r}_0 + \frac{2}{1 + \mathbf{g}^2} [\mathbf{g} \times \mathbf{r}_0 + \mathbf{g} \mathbf{g} \cdot \mathbf{r}_0] .$$

## EQUATIONS FOR VARIATION OF EULER'S PARAMETERS

It is also of interest to deduce the equations for the variation of elements in terms of Euler parameters:

$$\begin{aligned} \lambda_1 &= \sin \frac{1}{2} i \cos \frac{1}{2} (\omega - \Omega) , \\ \lambda_2 &= \sin \frac{1}{2} i \sin \frac{1}{2} (\omega - \Omega) , \\ \lambda_3 &= \cos \frac{1}{2} i \sin \frac{1}{2} (\omega + \Omega) , \\ \lambda_4 &= \cos \frac{1}{2} i \cos \frac{1}{2} (\omega + \Omega) . \end{aligned} \tag{68}$$

The additional constraint is

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1 . \tag{69}$$

Two of these parameters, namely  $\lambda_1$  and  $\lambda_2$ , were used by Hansen (Reference 6) in his lunar theory. The author has suggested the use of all four in the theory of artificial satellites (Reference 7) and in a modification of Hansen's lunar theory (Reference 16). Writing the Pfaffian (Equation 55) in the form

$$\phi = \sqrt{\mu a} dl + \sqrt{\mu a(1 - e^2)} \frac{d(\omega + \Omega) + \tan^2 \frac{i}{2} d(\omega - \Omega)}{1 + \tan^2 \frac{i}{2}} + F dt$$

and taking Equation 68 into account, we deduce

$$\phi = \sqrt{\mu a} dl + 2 \sqrt{\mu a(1 - e^2)} (\lambda_1 d\lambda_2 - \lambda_2 d\lambda_1 + \lambda_4 d\lambda_3 - \lambda_3 d\lambda_4) + F dt . \tag{70}$$

The Pfaffian equations associated with the  $\lambda$ -parameters take the form:

$$\frac{\partial \phi}{\partial \lambda_1} + 2d \left[ \sqrt{\mu a(1 - e^2)} \lambda_2 \right] - w \lambda_1 dt = 0 \quad (71)$$

$$\frac{\partial \phi}{\partial \lambda_2} - 2d \left[ \sqrt{\mu a(1 - e^2)} \lambda_1 \right] - w \lambda_2 dt = 0 \quad (72)$$

$$\frac{\partial \phi}{\partial \lambda_3} - 2d \left[ \sqrt{\mu a(1 - e^2)} \lambda_4 \right] - w \lambda_3 dt = 0 \quad (73)$$

$$\frac{\partial \phi}{\partial \lambda_4} + 2d \left[ \sqrt{\mu a(1 - e^2)} \lambda_3 \right] - w \lambda_4 dt = 0 , \quad (74)$$

where  $w$  is the Lagrangian multiplier associated with the constraint (Equation 69). Putting

$$A = na \sqrt{1 - e^2} \frac{da}{dt} - \frac{2na^2 e}{\sqrt{1 - e^2}} \frac{de}{dt} , \quad (75)$$

we can write Equations 71-74 in the form:

$$- A \lambda_2 - 4na^2 \sqrt{1 - e^2} \frac{d\lambda_2}{dt} - \frac{\partial F}{\partial \lambda_1} + w \lambda_1 = 0 , \quad (76)$$

$$+ A \lambda_1 + 4na^2 \sqrt{1 - e^2} \frac{d\lambda_1}{dt} - \frac{\partial F}{\partial \lambda_2} + w \lambda_2 = 0 , \quad (77)$$

$$+ A \lambda_4 + 4na^2 \sqrt{1 - e^2} \frac{d\lambda_4}{dt} - \frac{\partial F}{\partial \lambda_3} + w \lambda_3 = 0 , \quad (78)$$

$$- A \lambda_3 - 4na^2 \sqrt{1 - e^2} \frac{d\lambda_3}{dt} - \frac{\partial F}{\partial \lambda_4} + w \lambda_4 = 0 . \quad (79)$$

From this last set of equations and Equation 69 we have

$$A = - \lambda_2 \frac{\partial F}{\partial \lambda_1} + \lambda_1 \frac{\partial F}{\partial \lambda_2} - \lambda_3 \frac{\partial F}{\partial \lambda_4} + \lambda_4 \frac{\partial F}{\partial \lambda_3} , \quad (80)$$

and

$$w = B - 4na^2 \sqrt{1 - e^2} \left( - \lambda_1 \frac{d\lambda_2}{dt} + \lambda_2 \frac{d\lambda_1}{dt} + \lambda_3 \frac{d\lambda_4}{dt} - \lambda_4 \frac{d\lambda_3}{dt} \right) , \quad (81)$$

where, for the sake of brevity, we put

$$B = \lambda_1 \frac{\partial F}{\partial \lambda_1} + \lambda_2 \frac{\partial F}{\partial \lambda_2} + \lambda_3 \frac{\partial F}{\partial \lambda_3} + \lambda_4 \frac{\partial F}{\partial \lambda_4} . \quad (82)$$

From the Pfaffian equation

$$\frac{\partial \phi}{\partial e} = 0 ,$$

we deduce, after some transformations,

$$- \lambda_1 \frac{d\lambda_2}{dt} + \lambda_2 \frac{d\lambda_1}{dt} + \lambda_3 \frac{d\lambda_4}{dt} - \lambda_4 \frac{d\lambda_3}{dt} = - \frac{\sqrt{1-e^2}}{2na^2 e} \frac{\partial F}{\partial e} , \quad (83)$$

and Equation 81 becomes

$$w = B + \frac{2(1-e^2)}{e} \frac{\partial F}{\partial e} .$$

Substituting this value of  $w$  into Equations 76-79 we obtain:

$$\begin{aligned} \frac{d\lambda_1}{dt} &= + \frac{1}{4na^2 \sqrt{1-e^2}} \left( \frac{\partial F}{\partial \lambda_2} - \lambda_1 A - \lambda_2 B \right) - \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_2 \frac{\partial F}{\partial e} , \\ \frac{d\lambda_2}{dt} &= - \frac{1}{4na^2 \sqrt{1-e^2}} \left( \frac{\partial F}{\partial \lambda_1} + \lambda_2 A - \lambda_1 B \right) + \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_1 \frac{\partial F}{\partial e} , \\ \frac{d\lambda_3}{dt} &= - \frac{1}{4na^2 \sqrt{1-e^2}} \left( \frac{\partial F}{\partial \lambda_4} + \lambda_3 A - \lambda_4 B \right) + \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_4 \frac{\partial F}{\partial e} , \\ \frac{d\lambda_4}{dt} &= + \frac{1}{4na^2 \sqrt{1-e^2}} \left( \frac{\partial F}{\partial \lambda_3} - \lambda_4 A - \lambda_3 B \right) - \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_3 \frac{\partial F}{\partial e} . \end{aligned}$$

Taking Equations 80 and 82 into account we deduce from the last set of equations:

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \frac{1}{4na^2 \sqrt{1-e^2}} \left[ + (\lambda_4^2 + \lambda_3^2) \frac{\partial F}{\partial \lambda_2} \right. \\ &\quad \left. - (\lambda_1 \lambda_4 + \lambda_2 \lambda_3) \frac{\partial F}{\partial \lambda_3} - (\lambda_2 \lambda_4 - \lambda_1 \lambda_3) \frac{\partial F}{\partial \lambda_4} \right] - \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_2 \frac{\partial F}{\partial e} , \end{aligned}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = & \frac{1}{4na^2 \sqrt{1-e^2}} \left[ -(\lambda_3^2 + \lambda_4^2) \frac{\partial F}{\partial \lambda_1} \right. \\ & \left. - (\lambda_2 \lambda_4 - \lambda_1 \lambda_3) \frac{\partial F}{\partial \lambda_3} + (\lambda_1 \lambda_4 + \lambda_2 \lambda_3) \frac{\partial F}{\partial \lambda_4} \right] + \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_1 \frac{\partial F}{\partial e} , \end{aligned}$$

$$\begin{aligned} \frac{d\lambda_3}{dt} = & \frac{1}{4na^2 \sqrt{1-e^2}} \left[ +(\lambda_1 \lambda_4 + \lambda_2 \lambda_3) \frac{\partial F}{\partial \lambda_1} \right. \\ & \left. + (\lambda_2 \lambda_4 - \lambda_1 \lambda_3) \frac{\partial F}{\partial \lambda_2} - (\lambda_1^2 + \lambda_2^2) \frac{\partial F}{\partial \lambda_4} \right] + \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_4 \frac{\partial F}{\partial e} , \end{aligned}$$

$$\begin{aligned} \frac{d\lambda_4}{dt} = & \frac{1}{4na^2 \sqrt{1-e^2}} \left[ +(\lambda_2 \lambda_4 - \lambda_1 \lambda_3) \frac{\partial F}{\partial \lambda_1} \right. \\ & \left. - (\lambda_1 \lambda_4 + \lambda_2 \lambda_3) \frac{\partial F}{\partial \lambda_2} + (\lambda_1^2 + \lambda_2^2) \frac{\partial F}{\partial \lambda_3} \right] - \frac{\sqrt{1-e^2}}{2na^2 e} \lambda_3 \frac{\partial F}{\partial e} . \end{aligned}$$

A similar set of equations was recently deduced by the author in a less direct manner (Reference 16). From Equations 34, 75 and 80 we obtain

$$\frac{de}{dt} = + \frac{1-e^2}{na^2 e} \frac{\partial F}{\partial l} + \frac{\sqrt{1-e^2}}{2na^2 e} \left( \lambda_2 \frac{\partial F}{\partial \lambda_1} - \lambda_1 \frac{\partial F}{\partial \lambda_2} + \lambda_3 \frac{\partial F}{\partial \lambda_4} - \lambda_4 \frac{\partial F}{\partial \lambda_3} \right)$$

The disturbed position vector  $r$  is given by the formula:

$$r = \Gamma \cdot \begin{bmatrix} a(\cos u - e) \\ a \sqrt{1-e^2} \sin u \\ 0 \end{bmatrix} ,$$

where

$$u - e \sin u = l ,$$

where  $a, e, l$  are the disturbed elements. The expression for the matrix  $\Gamma$  in terms of  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  is given in Reference 7. We repeat it here for the sake of completeness:

$$\Gamma = [\lambda_{ij}] ,$$

$$\lambda_{11} = +\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2 ,$$

$$\lambda_{12} = -2(\lambda_1 \lambda_2 + \lambda_3 \lambda_4) ,$$

$$\begin{aligned}
\lambda_{13} &= + 2(\lambda_1 \lambda_3 - \lambda_2 \lambda_4) , \\
\lambda_{21} &= + 2(\lambda_3 \lambda_4 - \lambda_1 \lambda_2) , \\
\lambda_{31} &= + 2(\lambda_1 \lambda_3 + \lambda_2 \lambda_4) , \\
\lambda_{22} &= - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 + \lambda_4^2 , \\
\lambda_{32} &= + 2(\lambda_1 \lambda_4 - \lambda_2 \lambda_3) , \\
\lambda_{23} &= - 2(\lambda_1 \lambda_4 + \lambda_2 \lambda_3) , \\
\lambda_{33} &= - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 + \lambda_4^2 .
\end{aligned}$$

We can also form different sets of purely vectorial elements by taking Equation 15 as a basis. Writing Equation 15 in an equivalent form

$$\phi = - \sqrt{\mu a} \, dl + \sqrt{\mu a(1 - e^2)} \, \mathbf{P} \cdot d\mathbf{Q} - F \, dt \quad (84)$$

and putting

$$\begin{aligned}
\mathbf{p} &= \sqrt{\mu a} \, \mathbf{P} , \\
\mathbf{q} &= \sqrt{1 - e^2} \, \mathbf{Q} - l \mathbf{P} ,
\end{aligned}$$

we deduce from Equation 84

$$\phi = \mathbf{p} \cdot d\mathbf{q} - F \, dt .$$

Thus  $\mathbf{p}$  and  $\mathbf{q}$  represent a purely vectorial canonical set of elements. We have

$$\frac{d\mathbf{q}}{dt} = + \text{grad}_{\mathbf{p}} F ,$$

$$\frac{d\mathbf{p}}{dt} = - \text{grad}_{\mathbf{q}} F .$$

## CONCLUSION

The Pfaffian method is suggested for use in the search for new sets of elements and in the formation of equations for their variations. The application of vector analysis permits the formation of Pfaffian equations in a very compact and elegant form. A method of integration similar to von Zeipel's method (Reference 17) can be based on the expansion of the Pfaffian differential form in powers of a small parameter.

## REFERENCES

1. Bilimovich, A., "Über die Anwendungen der Pfaffschen Methode in der Störungstheorie," *Astr. Nachr.* 273:161-178, 1943.
2. Milankovich, M., "On Application of Vectorial Elements in the Computation of the Planetary Perturbations," *Bull. Acad. Math. Natur.* (A), Belgrade, No. 6, 1939 (In Serbian).
3. Cartan, E., "Leçons sur les Invariants Intégraux," Paris; A. Hermann et Fils, 1922.
4. Strömgren, B., *Pub. Med. Kobenhavns Obs.* No. 65, p. 5, 1929.
5. Musen, P., "On Strömgren's Method of Special Perturbations," *J. Astronaut. Sci.* 8:48-51, Summer 1961.
6. Hansen, P. A., "Fundamenta," Gotha: Carolum Glasser, p. 331, 1838.
7. Musen, P., "Application of Hansen's Theory to the Motion of an Artificial Satellite in the Gravitational Field of the Earth," *Journ. Geophys. Res.* 64(12):2271-2279, December 1959.
8. Herrick, S., "A Modification of the Variation of Constants Method for Special Perturbations," *Astron. Soc. Pacific Pub.* 60:321-323, October 1948.
9. Musen, P., "Special Perturbations of the Vectorial Elements," *Astron. J.* 59(7):262-267, August 1954.
10. Eckert, W. J., and Brouwer, D., "The Use of Rectangular Coordinates in the Differential Corrections of Orbits," *Astron. J.* 46(13):125-132, August 16, 1937.
11. Musen, P., and Carpenter, L., "On the General Planetary Perturbations in Rectangular Coordinates," *J. Geophys. Res.* 68(9):2727-2734, May 1, 1963.
12. Musen, P., "On the Long-Period Lunar and Solar Effects in the Motion of an Artificial Satellite, 2," *J. Geophys. Res.* 66(9):2797-2805, September 1961.
13. Cook, G. E., "Luni-Solar Perturbations of the Orbit of an Earth Satellite," Royal Aeronautical Establishment Report, Farnborough, 1961.
14. Kaula, W. M., "Development of the Lunar and Solar Disturbing Functions for a Close Satellite," *Astron. J.* 67(5):300-303, June 1962; also NASA Technical Note D-1126, January 1962.
15. Gibbs, J. W., "Vector Analysis," New York: Charles Scribner's Sons, 1901, pp. 343-345.
16. Musen, P., "On a Modification of Hansen's Lunar Theory," *J. Geophys. Res.* 68(5):1439-1456, March 1, 1963; also NASA Technical Note D-1745, June 1963.
17. von Zeipel, H., "Recherches sur le Mouvement des Petites Planètes," *Arkiv Matem. Astron. o Fysik*, 11(1):1-58, 1916-1917.

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